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Naked singularities are not singular in distorted gravity

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Abstract

We compute the Zero Point Energy (ZPE) induced by a naked singularity with the help of a reformulation of the Wheeler–DeWitt equation. A variational approach is used for the calculation with Gaussian Trial Wave Functionals. The one loop contribution of the graviton to the ZPE is extracted keeping under control the UltraViolet divergences by means of a distorted gravitational field. Two examples of distortion are taken under consideration: Gravity’s Rainbow and Noncommutative Geometry. Surprisingly, we find that the ZPE is no more singular when we approach the singularity.

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1. Introduction

Black holes are amazing astrophysical objects which are supposed to form as a consequence of a gravitational collapse of some matter field. An important black hole feature is the formation of a horizon preventing a far observer to see its own singularity. The simplest non-rotating and uncharged black hole can be represented by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2MG}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

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which is obtained by solving Einstein's Field Equations in vacuum. As one can easily see the horizon is located at $r_S = 2MG$. It is immediate to recognize that replacing M with $-M$, one obtains another solution of Einstein's Field Equations, but with a completely different structure: the singularity is no more protected by an event horizon and it is naked [1]. In 1969, Penrose suggested that there might be a sort of "cosmic censor" that forbids naked singularities from forming [2], namely singularities that are visible to distant observers. An immediate consequence of a negative Schwarzschild mass is that if one were to place two bodies initially at rest, one with a negative mass and the other with a positive mass, both would accelerate in the same direction going from the negative mass to the positive one. Furthermore if the two masses are of the same magnitude, they will uniformly accelerate forever. This means that a problem of classical stability emerges in such a geometry [3,4]. If, from one side, naked singularities are well examined from the classical point of view, it is non-trivial extracting information from the quantum point of view. Nevertheless, an interesting calculation is represented by the determination of Zero Point Energy (ZPE). It is important to remark that usually any attempt to perform a ZPE calculation inevitably faces UV divergences and therefore a regularization scheme is needed. One possible way to take under control such divergences is given by a zeta regularization. After the regularization a renormalization scheme can be adopted [5]. However, one can observe that at very high energies, it is likely that spacetime itself can be distorted by quantum fluctuations. The hope is that the distorted spacetime is also able to take under control the UV divergences. To this purpose, we will explore two proposals: *Gravity's Rainbow* and Noncommutative Geometry. When Gravity's Rainbow is taken under consideration, spacetime is endowed with two arbitrary functions $g_1(E/E_P)$ and $g_2(E/E_P)$ having the following properties

$$\lim_{E/E_P \rightarrow 0} g_1(E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P \rightarrow 0} g_2(E/E_P) = 1. \quad (2)$$

$g_1(E/E_P)$ and $g_2(E/E_P)$ appear into the solutions of the modified Einstein's Field Equations [6]

$$G_{\mu\nu}(E/E_P) = 8\pi G(E/E_P)T_{\mu\nu}(E/E_P) + g_{\mu\nu}\Lambda(E/E_P), \quad (3)$$

where $G(E/E_P)$ is an energy dependent Newton's constant, defined so that $G(0)$ is the low-energy Newton's constant and $\Lambda(E/E_P)$ is an energy dependent cosmological constant. Usually E is the energy associated with the particles deforming the spacetime geometry. Since the scale of deformation involved is the Planck scale, it is likely that spacetime itself fluctuates in such a way to produce a ZPE. However the deformed Einstein's gravity has only one particle available: the graviton. Therefore the particle probing the spacetime will be the graviton produced by the fluctuations of the spacetime itself. Note that the Rainbow's functions distort in different ways depending on the background. For example, for the Schwarzschild background one gets

$$ds^2 = -\left(1 - \frac{2MG(0)}{r}\right) \frac{dt^2}{g_1^2(E/E_P)} + \frac{dr^2}{\left(1 - \frac{2MG(0)}{r}\right)g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)}(d\theta^2 + \sin^2\theta d\phi^2), \quad (4)$$

where $G(0)$ is the low-energy Newton's constant. For the Friedmann–Lemaître–Robertson–Walker (FLRW) metric describing a homogeneous, isotropic and closed universe with line element, the Rainbow's Gravity distortion becomes [7–9]

$$ds^2 = -\frac{N^2(t)}{g_1^2(E/E_P)}dt^2 + \frac{a^2(t)}{g_2^2(E/E_P)}d\Omega_3^2, \quad (5)$$

where $N = N(t)$ is the lapse function taken to be homogeneous and $a(t)$ denotes the scale factor. Fixing our attention on the static case, we generalize the line element (4) in the following way

$$ds^2 = -\frac{N^2(r)}{g_1^2(E/E_P)}dt^2 + \frac{dr^2}{(1 - \frac{b(r)}{r})g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)}(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

where $N = N(r)$ is the lapse function and $b(r)$ is termed as the shape function and its range of variability depends on case to case. For example for the Schwarzschild metric we have $b(r) = 2MG$ and $r \in [r_t, +\infty)$, while for a naked singularity we have $b(r) = -2\bar{M}G$ and $r \in (0, +\infty)$. Of course, we are taking under consideration the simplest naked singularity. For example, one could also consider a Reissner–Nordström naked singularity or a Kerr naked singularity. However, the introduction of the charge in the former and rotation in the latter increase the technical level and momentarily they will not be considered. On the other hand, when a Noncommutative Geometry is taken under consideration, the spacetime is endowed with a commutator $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix which determines the fundamental discretization of spacetime. As shown in Ref. [12] and references therein, the classical Liouville measure

$$\frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3} \quad (7)$$

is distorted into

$$\frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3} \exp\left(-\frac{\theta}{4}k_i^2\right), \quad (8)$$

where k_i^2 is the radial wave number associated to each mode of the graviton. It is clear that the UV cut off θ is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. In a series of papers [11,15,16], we have applied the Gravity’s Rainbow formalism to the Zero Point Energy (ZPE) calculation and we have shown that appropriate choices of the Rainbow’s functions keep under control the UV divergences. The same finite result has been obtained in Ref. [12] with a Noncommutative Geometry. The key point is the following expectation value [20]¹

$$\frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{8\pi G}, \quad (9)$$

which is obtained by a formal manipulation of the Wheeler–DeWitt equation (WDW) [19]. Λ denotes the cosmological constant, while $\hat{\Lambda}_{\Sigma}$ is the operator containing all the information about the gravitational field. In this form, Eq. (9) can be used to compute ZPE provided that $\Lambda/8\pi G$ be considered as an eigenvalue of $\hat{\Lambda}_{\Sigma}$. Nevertheless, solving Eq. (9) is a quite impossible task, therefore we are oriented to use a variational approach with trial wave functionals. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case, are of the Gaussian type: this choice is justified by the fact that ZPE should be described by a good candidate of the “vacuum state”. However if we change the form of the wave functionals we also change the corresponding boundary conditions and therefore the description of the vacuum state. It is better to observe that the obtained eigenvalue $\Lambda/8\pi G$, is far to be a constant, rather

¹ An application of this calculation in the framework of Horáva–Lifshitz theory can be found in Ref. [21]. A slight variant of this calculation can be found in Ref. [22].

it will be dependent on some parameters which depend on the background under consideration. Therefore the correct interpretation is that of a “*dynamical cosmological constant*” evolving in r and M instead of a temporal parameter t . This is not a novelty, almost all the inflationary models try to substitute a cosmological constant Λ with some fields that change with time. In this case, it is the gravity itself that gives a dynamical aspect to the “*cosmological constant* Λ ”, or more correctly the ZPE $\Lambda/8\pi G$, without introducing any kind of external field but only quantum fluctuations of the *pure gravitational field*. Note that in this approach it will be the “*dynamical cosmological constant*” that will give information about the naked singularity. It is important to remark that we will not follow a collapsing star or a shell into a naked singularity, but we will consider a naked singularity which is already existing and motivated by the results obtained in Refs. [11,12,15,16], we would like to extend the same ZPE calculation to a naked singularity of the form

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{1 + \frac{2MG}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (10)$$

which, in the case of Gravity’s Rainbow will be distorted into the line element (6), while for the Noncommutative Geometry will remain same as described by Eq. (10). The starting point of our analysis will be the line element (10), which will also be our cornerstone of the whole paper which is organized as follows. In Section 2, we derive Eq. (9) and we extract the graviton one loop contribution to ZPE with respect to the desired background. In Section 3, we report the results of Ref. [11] with the help of the background (10) adapted for the naked singularity in a Gravity’s Rainbow environment. In Section 4, we report the results of Ref. [12] with the help of the background (10) adapted for the naked singularity in a Noncommutative environment. We summarize and conclude in Section 5. Units in which $\hbar = c = k = 1$ are used throughout the paper.

2. Setting up the ZPE calculation from the WDW equation

In this section we derive the general form for the ZPE calculation on a spherical symmetric background. The procedure relies heavily on the formalism outlined in Refs. [10–12]. The key point for the derivation is in the Arnowitt–Deser–Misner (\mathcal{ADM}) decomposition [13] of spacetime based on the following line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = (-N^2 + N_i N^i)dt^2 + 2N_j dt dx^j + g_{ij}dx^i dx^j, \quad (11)$$

where N is the *lapse* function and N_i the *shift* function. In terms of the \mathcal{ADM} variables, the four dimensional scalar curvature \mathcal{R} can be decomposed in the following way

$$\mathcal{R} = R + K_{ij}K^{ij} - (K)^2 - 2\nabla_\mu(Ku^\mu + a^\mu), \quad (12)$$

where

$$K_{ij} = -\frac{1}{2N}[\partial_t g_{ij} - N_{i|j} - N_{j|i}] \quad (13)$$

is the second fundamental form, $K = g^{ij}K_{ij}$ is its trace, R is the three dimensional scalar curvature and \sqrt{g} is the three dimensional determinant of the metric. The last term in (12) represents the boundary terms contribution where the four-velocity u^μ is the timelike unit vector normal to the spacelike hypersurfaces ($t = \text{constant}$) denoted by Σ_t and $a^\mu = u^\alpha \nabla_\alpha u^\mu$ is the acceleration of the timelike normal u^μ . Thus

$$\begin{aligned}\mathcal{L}[N, N_i, g_{ij}] &= \sqrt{-4}g(\mathcal{R} - 2\Lambda) \\ &= \frac{N}{2\kappa} \sqrt{g} [K_{ij}K^{ij} - K^2 + R - 2\Lambda - 2\nabla_\mu(Ku^\mu + a^\mu)]\end{aligned}\quad (14)$$

represents the gravitational Lagrangian density where $\kappa = 8\pi G$. After a Legendre transformation, the WDW equation simply becomes

$$\mathcal{H}\Psi = \left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}(R - 2\Lambda) \right] \Psi = 0, \quad (15)$$

where G_{ijkl} is the super-metric and where the conjugate super-momentum π^{ij} is defined as

$$\pi^{ij} = \frac{\delta \mathcal{L}}{\delta(\partial_t g_{ij})} = (g^{ij}K - K^{ij}) \frac{\sqrt{g}}{2\kappa}. \quad (16)$$

Note that $\mathcal{H} = 0$ represents the classical constraint which guarantees the invariance under time reparametrization. The other classical constraint represents the invariance by spatial diffeomorphism and it is described by $\pi_{|j}^{ij} = 0$, where the vertical stroke “|” denotes the covariant derivative with respect to the 3D metric g_{ij} . To reproduce Eq. (9) we have to multiply Eq. (15) by $\Psi^*[g_{ij}]$ and functionally integrate over the three spatial metric g_{ij} . Then by defining the volume of the hypersurface Σ as

$$V = \int_{\Sigma} d^3x \sqrt{g} \quad (17)$$

and

$$\hat{\Lambda}_{\Sigma} = (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}R/(2\kappa), \quad (18)$$

we arrive at

$$\frac{1}{V} \frac{\int \mathcal{D}[g_{ij}] \Psi^*[g_{ij}] \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} \Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}] \Psi^*[g_{ij}] \Psi[g_{ij}]} = -\frac{\Lambda}{8\pi G}, \quad (19)$$

namely Eq. (9). To further proceed, we consider

$$g_{ij} = \bar{g}_{ij} + h_{ij}, \quad (20)$$

where \bar{g}_{ij} is the background metric and h_{ij} is a quantum fluctuation around a background. However, to extract the graviton contribution, we also need an orthogonal decomposition on the tangent space of 3-metric deformations [14]

$$h_{ij} = \frac{1}{3}(\sigma + 2\nabla \cdot \xi)g_{ij} + (L\xi)_{ij} + h_{ij}^{\perp}. \quad (21)$$

The operator L maps the gauge vector ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3}g_{ij}(\nabla \cdot \xi), \quad (22)$$

h_{ij}^{\perp} is the traceless-transverse component of the perturbation (TT), namely

$$g^{ij}h_{ij}^{\perp} = 0, \quad \nabla^i h_{ij}^{\perp} = 0 \quad (23)$$

and h is the trace of h_{ij} . It is immediate to recognize that the trace element $\sigma = h - 2(\nabla \cdot \xi)$ is gauge invariant. If we perform the same decomposition also on the momentum π^{ij} , up to second order Eq. (19) becomes

$$\frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x [\hat{\Lambda}_{\Sigma}^{\perp} + \hat{\Lambda}_{\Sigma}^{\xi} + \hat{\Lambda}_{\Sigma}^{\sigma}]^{(2)} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}. \quad (24)$$

Concerning the measure appearing in (19), we have to note that the decomposition (21) induces the following transformation on the functional measure $\mathcal{D}h_{ij} \rightarrow \mathcal{D}h_{ij}^{\perp} \mathcal{D}\xi_i \mathcal{D}\sigma J_1$, where the Jacobian related to the gauge vector variable ξ_i is

$$J = \left[\det \left(\Delta g^{ij} + \frac{1}{3} \nabla^i \nabla^j - R^{ij} \right) \right]^{\frac{1}{2}}. \quad (25)$$

This is nothing but the famous Faddeev–Popov determinant. It becomes more transparent if ξ_a is further decomposed into a transverse part ξ_a^T with $\nabla^a \xi_a^T = 0$ and a longitudinal part ξ_a^{\parallel} with $\xi_a^{\parallel} = \nabla_a \psi$. Then J can be expressed by an upper triangular matrix for certain backgrounds (e.g. Schwarzschild in three dimensions). It is immediate to recognize that for an Einstein space in any dimension, cross terms vanish and J can be expressed by a block diagonal matrix. Since $\det AB = \det A \det B$, the functional measure $\mathcal{D}h_{ij}$ factorizes into

$$\mathcal{D}h_{ij} = (\det \Delta_V^T)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \Delta^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}} \mathcal{D}h_{ij}^{\perp} \mathcal{D}\xi^T \mathcal{D}\psi \quad (26)$$

leading to the Faddeev–Popov determinant with $(\Delta_V^{ij})^T = \Delta g^{ij} - R^{ij}$ acting on transverse vectors. Thus the inner product can be written as

$$\begin{aligned} & \int \mathcal{D}h_{ij}^{\perp} \mathcal{D}\xi^T \mathcal{D}\sigma \Psi^* [h_{ij}^{\perp}] \Psi^* [\xi^T] \Psi^* [\sigma] \Psi [h_{ij}^{\perp}] \Psi [\xi^T] \\ & \times \Psi [\sigma] (\det \Delta_V^T)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \Delta^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}}. \end{aligned} \quad (27)$$

Nevertheless, since there is no interaction between ghost fields and the other components of the perturbation at this level of approximation, the Jacobian appearing in the numerator and in the denominator simplify. The reason can be found in terms of connected and disconnected terms. The disconnected terms appear in the Faddeev–Popov determinant and above ones are not linked by the Gaussian integration. This means that disconnected terms in the numerator and the same ones appearing in the denominator cancel out. Therefore, (24) factorizes into three pieces. The piece containing E_{Σ}^{\perp} , the contribution of the transverse-traceless tensors (TT), is essentially the graviton contribution representing true physical degrees of freedom. Regarding the vector operator $\hat{\Lambda}_{\Sigma}^T$, we observe that under the action of infinitesimal diffeomorphism generated by a vector field ϵ_i , the components of (21) transform as follows [14]

$$\xi_j \longrightarrow \xi_j + \epsilon_j, \quad h \longrightarrow h + 2\nabla \cdot \epsilon, \quad h_{ij}^{\perp} \longrightarrow h_{ij}^{\perp}. \quad (28)$$

The Killing vectors satisfying the condition $\nabla_i \epsilon_j + \nabla_j \epsilon_i = 0$, do not change h_{ij} , and thus should be excluded from the gauge group. All other diffeomorphisms act on h_{ij} nontrivially. We need to fix the residual gauge freedom on the vector ξ_i . The simplest choice is $\xi_i = 0$. This new gauge fixing produces the same Faddeev–Popov determinant connected to the Jacobian J and therefore will not contribute to the final value. We are left with

$$\frac{1}{V} \frac{\langle \Psi^\perp | \int_\Sigma d^3x [\hat{A}_\Sigma^\perp]^{(2)} | \Psi^\perp \rangle}{\langle \Psi^\perp | \Psi^\perp \rangle} + \frac{1}{V} \frac{\langle \Psi^\sigma | \int_\Sigma d^3x [\hat{A}_\Sigma^\sigma]^{(2)} | \Psi^\sigma \rangle}{\langle \Psi^\sigma | \Psi^\sigma \rangle} = -\frac{\Lambda}{\kappa}. \quad (29)$$

Note that in the expansion of $\int_\Sigma d^3x \sqrt{g} R$ to second order, a coupling term between the TT component and scalar one remains. The scalar contribution \hat{A}_Σ^σ can be always gauged away by an appropriate choice of the vector field ϵ_j . Now that we have deduced the one loop approximation of Eq. (9), we need a regularization/renormalization process to keep under control the divergences. In the next section we will evaluate Eq. (29) distorted by Gravity's Rainbow which will be our regularization framework.

3. Setting up the ZPE computation with the Wheeler–DeWitt equation distorted by Gravity's Rainbow

In this section we derive how WDW modifies when the functions $g_1(E/E_P)$ and $g_2(E/E_P)$ distort the background (10). The form of the background is such that the *shift function*

$$N^i = -Nu^i = g_0^{4i} = 0 \quad (30)$$

vanishes, while N is the previously defined *lapse function*. Thus the definition of K_{ij} implies

$$K_{ij} = -\frac{\dot{g}_{ij}}{2N} = \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \tilde{K}_{ij}, \quad (31)$$

where the dot denotes differentiation with respect to the time t and the tilde indicates the quantity computed in absence of Rainbow's functions $g_1(E/E_P)$ and $g_2(E/E_P)$. The trace of the extrinsic curvature, therefore becomes

$$K = g^{ij} K_{ij} = g_1(E/E_P) \tilde{K} \quad (32)$$

and the momentum π^{ij} conjugate to the three-metric g_{ij} of Σ is

$$\pi^{ij} = \frac{\sqrt{g}}{2\kappa} (K g^{ij} - K^{ij}) = \frac{g_1(E/E_P)}{g_2(E/E_P)} \tilde{\pi}^{ij}. \quad (33)$$

Thus the distorted classical constraint becomes

$$\mathcal{H} = (2\kappa) \frac{g_1^2(E/E_P)}{g_2^3(E/E_P)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{2\kappa g_2(E/E_P)} \left(\tilde{R} - \frac{2\Lambda_c}{g_2^2(E/E_P)} \right) = 0, \quad (34)$$

where we have used the following property on R

$$R = g^{ij} R_{ij} = g_2^2(E/E_P) \tilde{R} \quad (35)$$

and where

$$G_{ijkl} = \frac{1}{2\sqrt{g}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}) = \frac{\tilde{G}_{ijkl}}{g_2(E/E_P)}. \quad (36)$$

The corresponding vacuum expectation value (9) becomes

$$\frac{g_2^3(E/E_P)}{\tilde{V}} \frac{\langle \Psi | \int_\Sigma d^3x \tilde{\Lambda}_\Sigma | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}, \quad (37)$$

with

$$\tilde{\Lambda}_\Sigma = (2\kappa) \frac{g_1^2(E/E_P)}{g_2^3(E/E_P)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}} \tilde{R}}{(2\kappa) g_2(E/E_P)}. \quad (38)$$

Extracting the TT tensor contribution from Eq. (37), we find

$$\begin{aligned} \hat{\Lambda}_\Sigma^\perp = & \frac{g_2^3(E/E_P)}{4\tilde{V}} \int_\Sigma d^3x \sqrt{\tilde{g}} \tilde{G}^{ijkl} \left[(2\kappa) \frac{g_1^2(E/E_P)}{g_2^3(E/E_P)} \tilde{K}^{-1\perp}(x, x)_{ijkl} \right. \\ & \left. + \frac{1}{(2\kappa) g_2(E/E_P)} (\tilde{\Delta}_L^m \tilde{K}^\perp(x, x))_{ijkl} \right], \end{aligned} \quad (39)$$

with the prescription that the corresponding eigenvalue equation transforms into the following way

$$(\hat{\Delta}_L^m h^\perp)_{ij} = E^2 h_{ij}^\perp \rightarrow (\tilde{\Delta}_L^m \tilde{h}^\perp)_{ij} = \frac{E^2}{g_2^2(E/E_P)} \tilde{h}_{ij}^\perp \quad (40)$$

in order to reestablish the correct way of transformation of the perturbation. Eq. (40) is the equation connecting the graviton energy with Gravity's Rainbow. The propagator $K^\perp(x, x)_{ijkl}$ will transform as

$$K^\perp(\vec{x}, \vec{y})_{ijkl} \rightarrow \frac{1}{g_2^4(E/E_P)} \tilde{K}^\perp(\vec{x}, \vec{y})_{ijkl}. \quad (41)$$

Thus the total one loop energy density for the graviton for the distorted GR becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2\tilde{V}} \sum_\tau g_1(E/E_P) g_2(E/E_P) \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right]. \quad (42)$$

The above expression makes sense only for $E_i^2(\tau) > 0$, where E_i are the eigenvalues of $\tilde{\Delta}_L^m$. With the help of Regge and Wheeler representation [23], the eigenvalue equation (40) can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r) \right] f_i(x) = \frac{E_{i,l}^2}{g_2^2(E/E_P)} f_i(x) \quad i = 1, 2, \quad (43)$$

where we have used reduced fields of the form $f_i(x) = F_i(x)/r$ and where we have defined two r -dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{3}{2r^2} b'(r) - \frac{3}{2r^3} b(r) \\ m_2^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{1}{2r^2} b'(r) + \frac{3}{2r^3} b(r) \end{cases} \quad (r \equiv r(x)). \quad (44)$$

In order to use the W.K.B. approximation, from Eq. (43) we can extract two r -dependent radial wave numbers

$$k_i^2(r, l, \omega_{i,nl}) = \frac{E_{i,nl}^2}{g_2^2(E/E_P)} - \frac{l(l+1)}{r^2} - m_i^2(r) \quad i = 1, 2. \quad (45)$$

To further proceed we use the W.K.B. method used by 't Hooft in the brick wall problem [24] and we count the number of modes with frequency less than ω_i , $i = 1, 2$. This is given approximately by

$$\tilde{g}(E_i) = \int_0^{l_{\max}} v_i(l, E_i)(2l+1)dl, \quad (46)$$

where $v_i(l, E_i)$, $i = 1, 2$ is the number of nodes in the mode with (l, E_i) , such that ($r \equiv r(x)$)

$$v_i(l, E_i) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \sqrt{k_i^2(r, l, E_i)}. \quad (47)$$

Here it is understood that the integration with respect to x and l_{\max} is taken over those values which satisfy $k_i^2(r, l, E_i) \geq 0$, $i = 1, 2$. With the help of Eqs. (46), (47), Eq. (42) leads to

$$\frac{\Lambda}{8\pi G} = -\frac{1}{\pi} \sum_{i=1}^2 \int_0^{+\infty} E_i g_1(E/E_P) g_2(E/E_P) \frac{d\tilde{g}(E_i)}{dE_i} dE_i. \quad (48)$$

This is the graviton contribution to the induced cosmological constant to one loop. The explicit evaluation of the density of states yields

$$\begin{aligned} \frac{d\tilde{g}(E_i)}{dE_i} &= \int \frac{\partial v(l, E_i)}{\partial E_i} (2l+1) dl \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} \frac{(2l+1)}{\sqrt{k_i^2(r, l, E)}} \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r) \right) dl \\ &= \frac{4}{3\pi} \int_{-\infty}^{+\infty} dx r^2 \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r) \right)^{\frac{3}{2}}. \end{aligned} \quad (49)$$

Plugging expression (49) into Eq. (48) and dividing for a volume factor, we obtain

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} E_i g_1(E/E_P) g_2(E/E_P) \frac{d}{dE_i} \sqrt{\left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r) \right)^3} dE_i, \quad (50)$$

where E^* is the value which annihilates the argument of the root. In the previous equation, we have included an additional 4π factor coming from the angular integration and we have assumed that the effective mass does not depend on the energy E . It is immediate to recognize that not every form of $g_1(E/E_P)$ and $g_2(E/E_P)$ can be used to compute the integrals in Eq. (50). Indeed, we need to impose that the Rainbow's functions satisfy convergence criteria. We fix our attention on the following choice

$$g_1(E/E_P) = \left(1 + \beta \frac{E}{E_P} \right) \exp\left(-\alpha \frac{E^2}{E_P^2} \right) \quad \text{and} \quad g_2(E/E_P) = 1, \quad (51)$$

which has been extensively used in Ref. [11]. For the Schwarzschild case, the background satisfies the following property

$$m_0^2(r) = m_2^2(r) = -m_1^2(r), \quad \forall r \in (r_t, r_1), \quad (52)$$

with $r_t = 2MG$ and $r \in [r_t, 5r_t/2]$ and for the dS, AdS and Minkowski background, the property

$$m_0^2(r) = m_2^2(r) = m_1^2(r), \quad \forall r \in (0, \infty) \quad (53)$$

is satisfied. So in the case of naked singularity, we find

$$m_1^2(r) = \frac{6}{r^2} + \frac{15\bar{M}G}{r^3} \quad (54)$$

and

$$m_2^2(r) = \frac{6}{r^2} + \frac{9\bar{M}G}{r^3}, \quad (55)$$

with $\bar{M} = -M$. Eq. (50) becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2}(I_1 + I_2), \quad (56)$$

where

$$I_{1,2} = \int_{\sqrt{m_{1,2}^2(r)}}^{\infty} E^2 \left(1 + \beta \frac{E}{E_P}\right) \exp\left(-\alpha \frac{E^2}{E_P^2}\right) \sqrt{E^2 - m_{1,2}^2(r)} dE. \quad (57)$$

Using the results of Appendix A, the integrals can be evaluated and we can finally write

$$\begin{aligned} \frac{\Lambda}{8\pi G} = & -\frac{E_P^4}{8\pi^2\alpha^2} \left[\exp(-x^2\alpha) \beta \sqrt{\pi} \frac{3+2x^2\alpha}{\sqrt{\alpha}} + 2\alpha x^2 \exp\left(-\frac{x^2\alpha}{2}\right) K_1\left(\frac{x^2\alpha}{2}\right) \right. \\ & \left. + \exp(-y^2\alpha) \beta \sqrt{\pi} \frac{3+2y^2\alpha}{\sqrt{\alpha}} + 2\alpha y^2 \exp\left(-\frac{y^2\alpha}{2}\right) K_1\left(\frac{y^2\alpha}{2}\right) \right], \end{aligned} \quad (58)$$

where

$$x = \sqrt{\frac{m_1^2(r)}{E_P^2}} = \frac{1}{rE_P} \sqrt{6 + \frac{15\bar{M}G}{r}} \quad \text{and} \quad y = \sqrt{\frac{m_2^2(r)}{E_P^2}} = \frac{1}{rE_P} \sqrt{6 + \frac{9\bar{M}G}{r}}. \quad (59)$$

It is useful to see what happens when x and $y \rightarrow \infty$ in Eq. (58). Taking the leading term, one gets

$$\frac{\Lambda}{8\pi G E_P^4} = -\frac{\beta}{2\pi^{3/2}\alpha^{3/2}} [e^{-x^2\alpha} x^2 + e^{-y^2\alpha} y^2], \quad (60)$$

while when x and $y \rightarrow 0$, we find

$$\begin{aligned} \frac{\Lambda}{8\pi G E_P^4} = & -\frac{4+3\sqrt{\pi/\alpha}\beta}{4\pi^2\alpha^2} + \frac{2+\sqrt{\pi/\alpha}\beta}{8\pi^2\alpha} (x^2 + y^2) - \frac{x^4}{32\pi^2} \log\left(\frac{x^4\alpha^2 e^{1+2\gamma-2\sqrt{\pi/\alpha}\beta}}{16}\right) \\ & - \frac{y^4}{32\pi^2} \log\left(\frac{y^4\alpha^2 e^{1+2\gamma-2\sqrt{\pi/\alpha}\beta}}{16}\right), \end{aligned} \quad (61)$$

where γ is Euler's constant. It is immediate to see that if we set

$$\beta = -\frac{4}{3}\sqrt{\frac{\alpha}{\pi}}, \quad (62)$$

then

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{\Lambda}{8\pi G E_P^4} = -\frac{4 + 3\sqrt{\pi/\alpha} \beta}{4\pi^2 \alpha^2} = 0. \quad (63)$$

Using the explicit form of the variables x and y of Eq. (59) and plugging the value of the parameter β found in Eq. (62) into the asymptotic expansion (60), one finds that the leading term behaves as

$$\frac{\Lambda}{8\pi G E_P^4} = \frac{2}{3\pi^2 \alpha r^3 E_P^2} \left[15\bar{M}G \exp\left(-\frac{\alpha 15\bar{M}G}{r^3 E_P^2}\right) + 9\bar{M}G \exp\left(-\frac{\alpha 9\bar{M}G}{r^3 E_P^2}\right) \right], \quad (64)$$

where it is meant that either $r \rightarrow 0$ or $\bar{M} \rightarrow \infty$. Nevertheless, the case in which $\bar{M} \rightarrow \infty$ is unphysical because it represents a singularity which fills the whole universe. On the other hand the case in which $r \rightarrow 0$ represents a naked singularity which is no more singular. In other words the distortion due to Gravity's Rainbow can cure also the problem of the singularity which appears appropriate for a correct theory of Quantum Gravity. Note that the regularity at the origin has been obtained also for the de Sitter and anti-de Sitter spaces in Ref. [11]. It is also important to remark that the regularity at $r = 0$, does not appear for every proposal of the Rainbow's functions $g_1(E/E_P)$ and $g_2(E/E_P)$. Indeed the proposal

$$g_1\left(\frac{E}{E_P}\right) = \left(1 + c_2 \frac{E}{E_P}\right) \exp\left(-c_1 \frac{E^2}{E_P^2}\right) \quad g_2(E/E_P) = 1 + c_3 \frac{E}{E_P} \quad (65)$$

discussed in Ref. [16] cannot be adopted here because in the trans-Planckian region the argument of the integrals (57) become independent on the radial variable and therefore they are not suppressed near the singularity. Therefore it appears that the choice (51) is very special in this context. In the next section, we will compute the effect of a Noncommutative theory on a naked singularity background.

4. Setting up the ZPE computation with the Wheeler–DeWitt equation distorted by a Noncommutative Geometry

If we avoid the use of Gravity's Rainbow and we introduce a Noncommutative Geometry, the first thing to do is the recovery of the one loop contribution of the graviton

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2V} \sum_{\tau} \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right]. \quad (66)$$

Eq. (66) has the same expression of Eq. (42), but with $g_1(E/E_P) = g_2(E/E_P) = 1$. However, to obtain a finite result we need to replace the classical Liouville counting number of nodes

$$dn = \frac{d^3 \vec{x} d^3 \vec{k}}{(2\pi)^3} \quad (67)$$

with [17,18]

$$dn_i = \frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3} \exp\left(-\frac{\theta}{4}k_i^2\right), \quad (68)$$

where

$$k_i^2 = E_{i,nl}^2 - m_i^2(r) \quad i = 1, 2. \quad (69)$$

This deformation corresponds to an effective cut off on the background geometry (10). The UV cut off is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. The virtue of this kind of deformation is its exponential damping profile, which encodes an intrinsic nonlocal character into fields $f_i(x)$. Plugging (47) into (46) and taking account of (68), the number of modes with frequency less than ω_i , $i = 1, 2$ is given by

$$g(E_i) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} \sqrt{E_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r)} (2l+1) \exp\left(-\frac{\theta}{4}k_i^2\right) dl. \quad (70)$$

After integration over modes, one gets

$$g(E_i) = \frac{2}{3\pi} \int_{-\infty}^{+\infty} dx r^2 \left[\frac{3}{2} \sqrt{(E_{i,nl}^2 - m_i^2(r))^3} \exp\left(-\frac{\theta}{4}(E_{i,nl}^2 - m_i^2(r))\right) \right]. \quad (71)$$

This form of $g(E_i)$ allows an integration by parts in (42) leading to

$$\frac{\Lambda}{8\pi G} = -\frac{1}{4\pi^2 V} \sum_{i=1}^2 \int_0^{+\infty} E_i \frac{dg(E_i)}{dE_i} dE_i = \frac{1}{4\pi^2 V} \sum_{i=1}^2 \int_0^{+\infty} g(E_i) dE_i. \quad (72)$$

This is the graviton contribution to the induced cosmological constant at one loop, where an additional 4π coming from the angular integration has been included. Plugging Eq. (71) into Eq. (72), one finds

$$\frac{\Lambda}{8\pi G} = \frac{1}{6\pi^2} \sum_{i=1}^2 \int_{\sqrt{m_i^2(r)}}^{+\infty} \sqrt{(\omega^2 - m_i^2(r))^3} e^{-\frac{\theta}{4}(\omega^2 - m_i^2(r))}, \quad (73)$$

where $m_i^2(r)$ are the effective masses described by Eqs. (54), (55).

Plugging the result of (B.3) into (72), we get

$$\begin{aligned} \frac{\Lambda}{8\pi G} = \frac{1}{12\pi^2} \left(\frac{4}{\theta}\right)^2 & \left[\left(\frac{1}{2} z(1-z) K_1\left(\frac{z}{2}\right) + \frac{1}{2} z^2 K_0\left(\frac{z}{2}\right) \right) \exp\left(\frac{z}{2}\right) \right. \\ & \left. + \left(\frac{1}{2} w(1-w) K_1\left(\frac{w}{2}\right) + \frac{1}{2} w^2 K_0\left(\frac{w}{2}\right) \right) \exp\left(\frac{w}{2}\right) \right] \end{aligned} \quad (74)$$

where

$$\begin{cases} z = m_1^2(r) \frac{\theta}{4} = \left(\frac{6}{r^2} + \frac{15\bar{M}G}{r^3} \right) \frac{\theta}{4} \\ w = m_2^2(r) \frac{\theta}{4} = \left(\frac{6}{r^2} + \frac{9\bar{M}G}{r^3} \right) \frac{\theta}{4} \end{cases}. \quad (75)$$

To analyze these results we consider the asymptotic expansion for $z, w \rightarrow \infty$ which means $r \ll \sqrt{\theta}$. Then one gets

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{12\pi^2} \left(\frac{4}{\theta}\right)^2 \frac{3}{8} \left(\sqrt{\frac{\pi}{z}} + \sqrt{\frac{\pi}{w}}\right) \rightarrow 0. \quad (76)$$

This corresponds to the correct behavior in a spacetime region where the curvature vanishes. On the other hand, for $r \gg \sqrt{\theta}$ we have $z, w \rightarrow 0$ which implies

$$\begin{aligned} \frac{\Lambda}{8\pi G} \simeq \frac{1}{12\pi^2} \left(\frac{4}{\theta}\right)^2 & \left[2 - \frac{z+w}{2} - \frac{3}{8} \ln\left(\frac{ze^{\gamma+\frac{7}{6}}}{4}\right) z^2 \right. \\ & \left. - \frac{3}{8} \ln\left(\frac{we^{\gamma+\frac{7}{6}}}{4}\right) w^2 \right] \rightarrow \frac{8}{3\pi^2 \theta^2} \end{aligned} \quad (77)$$

i.e. a finite value of the cosmological term.

5. Conclusions

In this paper we have considered the ZPE contribution deriving from an existing naked singularity. As an example we have considered the negative Schwarzschild mass which is the simplest model of naked singularity. We have computed the ZPE to one loop which is described as an induced cosmological constant. To keep under control the UV divergencies, instead of using a standard regularization/renormalization scheme we have chosen to distort the gravitational field with the help of two proposals: Gravity's Rainbow and the Noncommutative Geometry. This choice has been motivated by several results obtained computing the ZPE on some spherically symmetric background [11,12,15,16]. What we have found is that the distortion of the gravitational field can eliminate the singularity. In particular, we find that some Rainbow's functions suppress the divergent behavior so strongly in such a way to give regularity even to the point $r = 0$. Of course this cannot happen for every choice of the Rainbow's functions, as shown with the choice (65). It is important to remark that the choice of $g_1(E/E_P)$ and $g_2(E/E_P)$ for a ZPE calculation is restricted by the condition (2) and by the condition that the integrals for the graviton to one loop be finite. This means that the choice is not unique. Indeed in Ref. [15], the adopted choice was

$$\frac{g_1(E/E_P)}{g_2(E/E_P)} = \exp\left(-\frac{E}{E_P}\right), \quad (78)$$

while in this paper, inspired by Noncommutative Geometry, we have chosen

$$\frac{g_1(E/E_P)}{g_2(E/E_P)} = \left(1 + \beta \frac{E}{E_P}\right) \exp\left(-\alpha \frac{E^2}{E_P^2}\right), \quad (79)$$

and both the choices lead to a finite result. Nothing prevents to relax the condition (79) into condition (78), but this goes beyond the purpose of this paper. It is important to remark that in Gravity's Rainbow with the choice (51), the Minkowski limit test is satisfied. We draw to the reader's attention that for Minkowski limit we mean the following prescription [16]

$$\lim_{M \rightarrow 0} \frac{\Lambda}{8\pi G} = 0. \quad (80)$$

This means that when the background is switched off, one should recover the features of a Minkowski background. Note that the same test is not passed by a Noncommutative distortion. Indeed looking at Eq. (77), one finds

$$\lim_{M \rightarrow 0} \frac{\Lambda}{8\pi G} = \frac{8}{3\pi^2 \theta^2}, \quad (81)$$

namely the granularity of the Noncommutative Geometry persists independently on the value of the naked singularity. Only when $\theta \rightarrow \infty$, we have the vanishing limit, but this is a unphysical situation and therefore will be discarded. One possibility to overcome this difficulty is in a further modification of the theory coming from the replacement of the 4D scalar curvature \mathcal{R} with an arbitrary function of the scalar curvature, namely an $f(\mathcal{R})$ theory. Actually, one could introduce complicated combinations including \mathcal{R}^2 , $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, $\mathcal{R}^{\mu\nu\alpha\beta}\mathcal{R}_{\mu\nu\alpha\beta}$, $\mathcal{R} \square \mathcal{R}$, $\mathcal{R} \square^k \mathcal{R}$. These modifications are known under the name of Extended Theories of Gravity (ETG) and they have been introduced to explain data on the Large Scale Structure of Space–Time [25]. Since ETG introduce higher curvature terms, we have a benefit even at short scales where the construction of an effective action in Quantum Gravity is possible [26]. It is interesting to note that combining the simplest ETG model, namely an $f(\mathcal{R})$ theory with Gravity’s Rainbow, one obtains a model with interesting features in the Infra-Red and which is finite in the Ultra-Violet range, at least to one loop [16]. Moreover, thanks to the flexibility of the $f(\mathcal{R})$ term one can obtain the appropriate Minkowski limit, in the sense of Eq. (80). It is interesting to observe that the same behavior is present for the Schwarzschild solution. Therefore it seems that the ZPE calculation in the context of naked singularity with a Gravity’s Rainbow distortion appears to be special.

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Appendix A. Integrals for Gravity’s Rainbow distortion

In this appendix we give the rules to solve the integrals I_1 and I_2 , given by Eq. (57) with

$$g_1\left(\frac{E}{E_P}\right) = \left(1 + \beta \frac{E}{E_P}\right) \exp\left(-\alpha \frac{E^2}{E_P^2}\right), \quad g_2\left(\frac{E}{E_P}\right) = 1; \quad \alpha > 0, \beta \in \mathbb{R}. \quad (A.1)$$

Changing variables $E = \sqrt{x}$, the first term of the integral in (57) becomes

$$\begin{aligned} I &= \frac{1}{2} \int_{\sqrt{m^2}}^{\infty} \exp\left(-\alpha \frac{x}{E_P^2}\right) \sqrt{x} \sqrt{x - m^2} dx \\ &= \frac{E_P^4}{2\sqrt{\pi}} \left(\frac{m^2}{\alpha E_P^2}\right) \Gamma\left(\frac{3}{2}\right) \exp\left(-\frac{\alpha m^2}{2E_P^2}\right) K_1\left(\frac{\alpha m^2}{2E_P^2}\right), \end{aligned} \quad (A.2)$$

where we have used the following relationship

$$\int_u^{\infty} (x-u)^{\mu-1} x^{\mu-1} \exp(-\beta x) dx = \frac{\Gamma(\mu)}{\sqrt{\pi}} \left(\frac{u}{\beta}\right)^{\mu-1/2} \exp\left(-\frac{\beta u}{2}\right) K_{\mu-1/2}\left(\frac{\beta u}{2}\right) \quad (A.3)$$

$\text{Re } \mu > 0, \text{ Re } \beta u > 0$

and we have momentarily suppressed the suffices 1, 2. The second term of the integral in (57) becomes

$$\begin{aligned} I_\beta &= \int_{\sqrt{m^2}}^{\infty} \exp\left(-\alpha \frac{E^2}{E_P^2}\right) \frac{E^3}{E_P} \sqrt{E^2 - m^2} dE \frac{1}{2E_P} \int_{\sqrt{m^2}}^{\infty} \exp\left(-\alpha \frac{x}{E_P^2}\right) x \sqrt{x - m^2} dx \\ &= \frac{\sqrt{\pi} E_P^4}{4\alpha^{5/2}} \left(3 + \frac{2\alpha}{E_P^2} m^2\right) \exp\left(-\frac{\alpha m^2}{E_P^2}\right), \end{aligned} \quad (\text{A.4})$$

where we have used the following relationship

$$\int_a^{\infty} dx (x - a)^{1/2} x \exp(-\mu x) = \frac{\sqrt{\pi}}{4} \mu^{-5/2} (3 + 2\mu a) \exp(-\mu a) \quad a > 0, \mu > 0. \quad (\text{A.5})$$

Appendix B. Integrals for Noncommutative Geometry distortion

In this appendix, we explicitly compute the integrals coming from (72). We begin with

$$\begin{aligned} &\int_{\sqrt{m_0^2(r)}}^{+\infty} \sqrt{(\omega^2 - m_0^2(r))^3} e^{-\frac{\theta}{4}(\omega^2 - m_0^2(r))} d\omega \\ &\stackrel{\omega^2=x}{=} \frac{1}{2} \int_{\sqrt{m_0^2(r)}}^{+\infty} \sqrt{(x - m_0^2(r))^3} e^{-\frac{\theta}{4}(x - m_0^2(r))} \frac{dx}{\sqrt{x}} \\ &= \exp\left(\frac{m_0^2(r)\theta}{4}\right) \frac{1}{2} \left(\frac{\theta}{4}\right)^{-\frac{3}{2}} \sqrt{m_0^2(r)} \Gamma\left(\frac{5}{2}\right) \\ &\quad \times \exp\left(-\frac{m_0^2(r)\theta}{8}\right) W_{-1,-1}\left(\frac{m_0^2(r)\theta}{4}\right), \end{aligned} \quad (\text{B.1})$$

where we have used the following relationship

$$\begin{aligned} &\int_u^{+\infty} x^{v-1} (x - u)^{\mu-1} e^{-\beta x} dx = \beta^{-\frac{v+\mu}{2}} u^{\frac{v+\mu-2}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) W_{\frac{v-\mu}{2}, \frac{1-v-\mu}{2}}(\beta u) \\ &\quad \text{Re } \mu > 0, \text{ Re } \beta u > 0, \end{aligned} \quad (\text{B.2})$$

where $W_{\mu,\nu}(x)$ is the Whittaker function and $\Gamma(\nu)$ is the gamma function. Further manipulation on (B.1) leads to

$$\frac{1}{2} \left(\frac{\theta}{4}\right)^{-2} \left(\frac{1}{2} x(1-x) K_1\left(\frac{x}{2}\right) + \frac{1}{2} x^2 K_0\left(\frac{x}{2}\right)\right) \exp\left(\frac{x}{2}\right), \quad (\text{B.3})$$

where

$$x = \frac{m_0^2(r)\theta}{4}. \quad (\text{B.4})$$

It is useful to write an asymptotic expansion for $K_0(\frac{x}{2})$ and $K_1(\frac{x}{2})$. We get

$$\begin{aligned} K_0\left(\frac{x}{2}\right) &\simeq \sqrt{\pi} e^{-\frac{x}{2}} x^{-\frac{1}{2}} \left(1 - \frac{1}{4x}\right) + O\left(x^{-\frac{5}{2}}\right) \\ K_1\left(\frac{x}{2}\right) &\simeq \sqrt{\pi} e^{-\frac{x}{2}} x^{-\frac{1}{2}} \left(1 + \frac{3}{4x}\right) + O\left(x^{-\frac{5}{2}}\right). \end{aligned} \quad (\text{B.5})$$

Plugging expansion (B.5) into expression (B.3), one obtains that the asymptotic behavior is given by

$$\frac{1}{2} \left(\frac{\theta}{4}\right)^{-2} \sqrt{\pi} \left(\frac{1}{2} \sqrt{x} (1-x) \left(1 + \frac{3}{4x}\right) + \frac{1}{2} \sqrt{x^3} \left(1 - \frac{1}{4x}\right)\right) + O(x^{-\frac{5}{2}}) \quad (\text{B.6})$$

and after a further simplification, one gets

$$\frac{1}{2} \left(\frac{\theta}{4}\right)^{-2} \frac{3}{8} \sqrt{\frac{\pi}{x}} \quad (\text{B.7})$$

while when $x \rightarrow 0$, one gets

$$\frac{1}{2} \left(\frac{\theta}{4}\right)^{-2} \left[1 - \frac{x}{2} + \left(-\frac{7}{16} - \frac{3}{8} \ln\left(\frac{x}{4}\right) - \frac{3}{8} \gamma\right) x^2\right]. \quad (\text{B.8})$$

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